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Oriented Percolation in One–dimensional $1/|x-y|^2$ Percolation Models

D.H.U. Marchetti · V. Sidoravicius · M.E. Vares

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Abstract We consider independent edge percolation models on $\mathbb{Z}$, with edge occupation probabilities

$$p_{\{x,y\}} = \begin{cases} p & \text{if } |x-y| = 1, \\ 1 - \exp\{-\beta/|x-y|^2\} & \text{otherwise.} \end{cases}$$

We prove that oriented percolation occurs when $\beta > 1$ provided p is chosen sufficiently close to 1, answering a question posed in Newman and Schulman (Commun. Math. Phys. 104:547, 1986). The proof is based on multi-scale analysis.

Keywords Long-range edge percolation model · Oriented percolation · Multi-scale analysis · Dynamical renormalization

1 Introduction

It is well known that $1/r^2$ gives the "critical" falloff for percolation in one-dimensional long range independent edge percolation models. Moreover, for the one dimensional Fortuin–Kasteleyn (FK) random cluster model with weighting factor $\kappa \geq 1$ and edge occupation

D.H.U. Marchetti (✉)
Instituto de Física, Universidade de São Paulo, Caixa Postal 66318, 05314-970 São Paulo, SP, Brasil
e-mail: marchett@if.usp.br

V. Sidoravicius
Centrum Wiskunde & Informatica (CWI), Science Park 123, 1098 XG Amsterdam, Netherlands

V. Sidoravicius
Instituto de Matemática Pura e Aplicada (IMPA), Estrada Dona Castorina 110, Jardim Botânico, 22460-320 Rio de Janeiro, RJ, Brasil
e-mail: vladas@impa.br

M.E. Vares
Centro Brasileiro de Pesquisas Físicas (CBPF), Rua Dr. Xavier Sigaud 150, 22290-180 Rio de Janeiro, RJ, Brasil
e-mail: eulalia@cbpf.br

probabilities of the form $p_{\{x,y\}} = f(|x-y|)$, with $r^2 f(r) \to \beta > 0$ as $r \to +\infty$, it is known that for fixed $f(j) < 1$, $j \geq 2$ and varying $p = f(1)$, the value $\beta^* = 1$ is critical in the sense that for $\beta \leq 1$ percolation cannot occur unless $p = 1$ (see [1]), while for $\beta > 1$ there is percolation provided p is sufficiently close to one (see [11] and [14]). Such results are important in the description of the phase transition diagram for the one-dimensional long range Ising models studied earlier by Fröhlich and Spencer in [8] and for the corresponding Potts models [2, 11, 14], as these spin systems can be constructed by a random coloring of the clusters in the FK model with $\kappa = 2$, or $\kappa > 2$ integer, respectively. For the particular case of independent edge percolation models ($\kappa = 1$) earlier results were obtained in [16], where it was proven that $\beta^* \leq 1$ in this case, and that *oriented percolation* occurs when $\lim_{x\to\infty} x^s f(x) > 0$ for some $1 < s < 2$. The question whether oriented percolation occurs in the boundary case $s = 2$ remained unanswered. Theorem 1.1 below gives an affirmative answer; the result is stated for the particular example of edge probabilities in (1.1) below, and oriented percolation is shown when $\beta > 1$ and $p < 1$ is sufficiently close to one. The proofs can be easily adapted to include any $f(\cdot)$ satisfying $\lim_{x\to\infty} x^2 f(x) > 1$. In this sense $\beta^* = 1$ remains critical also for oriented percolation.

Our main result (Theorem 1.1) deals with the independent percolation model. On the other hand, known FKG inequalities and the above mentioned representation yield at once an application to the long range Ising (and Potts) models, which we state as Corollary 1.2. This is the reason for the preliminaries on this more general context of FK measures.

Preliminaries Consider the infinite complete graph with set of vertices $\mathcal{V} = \mathbb{Z}$ and set of edges $\mathbb{E} = \{\{x,y\},\ x \neq y,\ x, y \in \mathbb{Z}\}$, and let $\Omega = \{0,1\}^{\mathbb{E}}$. One-dimensional long-range FK random cluster models with weighting parameter $\kappa \geq 1$ are probability measures on $\sigma(\Omega)$, the usual product σ-algebra on Ω. To define them, let us first fix ν the Bernoulli product measure on Ω, with $\nu(\omega_{\{x,y\}} = 1) = p_{\{x,y\}}$ given by

$$p_{\{x,y\}} = \begin{cases} p & \text{if } |x-y| = 1, \\ 1 - \exp\{-\frac{\beta}{|x-y|^2}\} & \text{otherwise,} \end{cases} \tag{1.1}$$

where $0 < p < 1$ and $\beta > 0$ are fixed parameters.

Notation We write $q_{\{x,y\}} = 1 - p_{\{x,y\}}$; for $e = \{x,y\}$ we will write p_e instead of $p_{\{x,y\}}$, and say that e "is open" if $\omega_e = 1$. The length of an edge $e = \{x,y\}$ is $|x-y|$.

Finite Volume FK Measures Given $I \subset \mathbb{Z}$, consider $\mathbb{E}(I) = \{\{x,y\} \in \mathbb{E} \colon x, y \in I\}$, $\Omega_I = \{0,1\}^{\mathbb{E}(I)}$ and $\bar{\Omega}_I = \{0,1\}^{\mathbb{E}\setminus\mathbb{E}(I^c)}$, where $I^c = \mathbb{Z}\setminus I$. Assume that $|I| < \infty$. The corresponding finite volume *free FK-measure* is the probability measure $\mu^f_{\kappa,I}$ on Ω_I given by

$$\mu^f_{\kappa,I}(A) = \frac{\int_A \kappa^{\mathcal{C}_I(\omega)} \nu_I(d\omega)}{\int_{\Omega_I} \kappa^{\mathcal{C}_I(\omega)} \nu_I(d\omega)}, \quad A \subset \Omega_I, \tag{1.2}$$

where ν_I is the restriction of ν to Ω_I, and $\mathcal{C}_I(\omega)$ denotes the number of disjoint connected components in the graph determined by $\omega \in \Omega_I$ (i.e. the graph with vertices in I whose edges coincide with those e such that $\omega_e = 1$). The corresponding *wired FK-measure* $\mu^w_{\kappa,I}$ is a probability measure on $\bar{\Omega}_I$, defined similarly as in (1.2), replacing ν_I by $\bar{\nu}_I$, the restriction of ν to $\bar{\Omega}_I$ (so that $A \in \sigma(\bar{\Omega}_I)$ the usual product sigma algebra), and $\mathcal{C}_I(\omega)$ by $\bar{\mathcal{C}}_I(\omega)$, the number of disjoint connected components intersecting I in the graph with vertices in $\mathbb{Z}$ determined by $\bar{\omega}$, the configuration which extends $\omega \in \bar{\Omega}_I$ by setting $\bar{\omega}_e = 1$ for all $e \in$

$\mathbb{E}(I^c)$. Thus we may see $\mu^w_{\kappa,I}$ as a measure on Ω, concentrated on the configurations for which all edges in $\mathbb{E}(I^c)$ are open. Analogously, we may think of $\mu^f_{\kappa,I}$ as a probability measure on Ω, concentrated on the configurations ω such that $\omega_e = 0$ for any $e \in \mathbb{E} \setminus \mathbb{E}(I)$. Keeping this in mind we have the following well-known property.

The Infinite Volume Limit On Ω we consider the usual partial order: $\omega \le \omega'$ if $\omega_e \le \omega'_e$ for each $e \in \mathbb{E}$. By the FKG inequality (see [2, 5]), one has

$$\mu^f_{\kappa,I}(g) \le \mu^f_{\kappa,I'}(g) \le \mu^w_{\kappa,I'}(g) \le \mu^w_{\kappa,I}(g)$$

for any finite intervals $I \subset I' \subset \mathbb{Z}$, and any non-decreasing continuous function $g : \Omega \to \mathbb{R}$. Thus, as $I \nearrow \mathbb{Z}$ the limit measures μ^f_κ and μ^w_κ exist. Moreover, $\mu^f_\kappa \le \mu^w_\kappa$ in FKG sense.[1] If $\kappa = 1$, trivially $\mu^f_\kappa = \mu^w_\kappa = \nu$. Since $p_{\{x,y\}} \equiv f(|x-y|)$ the measures μ^f_κ and μ^w_κ are translation invariant; both are ergodic.

For a more general and complete discussion on the construction of random cluster measures, including issues in the infinite volume limit for general external conditions, see e.g. [9, 10] (focused mostly in short range models). This is particularly delicate when $0 < \kappa < 1$. Fix $\omega \in \Omega$. An alternating sequence of vertices and edges $x = x_1, e_1, x_2, \dots, x_{n-1}, e_{n-1}, x_n = y$, $n \ge 1$, is called a path connecting x to y, and we say that the path is open if $\omega_{e_i} \equiv \omega_{\{x_i,x_{i+1}\}} = 1$, $1 \le i \le n-1$. We say that $C \subset \mathbb{Z}$ is connected if for any two distinct vertices x, y in C there exists an open path π connecting them. A maximal connected set is called an *open cluster*, and $C_x(\omega)$ denotes the open cluster containing $x \in \mathbb{Z}$ (we write $C_x(\omega) = \{x\}$ if $\omega_{\{x,y\}} = 0$, for all $y \in \mathbb{Z} \setminus \{x\}$). A path $\pi = (x_1, \dots, x_n)$ connecting x to y, $x < y$, is called *oriented* if $x_1 = x < x_2 < \cdots < x_{n-1} < x_n = y$, and we write $x \rightsquigarrow y$ when there is an open oriented path connecting x to y. Analogously we define $C^+_x = \{y : x \rightsquigarrow y\}$, and the event

$$[x \rightsquigarrow \infty] = [|C^+_x| = \infty].$$

We are ready to state our main result.

Theorem 1.1 *For any* $\beta > 1$, *there exist* $0 < p_0 < 1$ *such that, if* $p > p_0$, *then*

$$\nu(0 \rightsquigarrow \infty) \ge 1 - \epsilon \tag{1.3}$$

holds with $\epsilon = \epsilon(p) \searrow 0$ *as* $p \nearrow 1$.

Remark 1 Let $\kappa > 1$. The statements in Theorem 4.1 of [2] imply that $\nu \le \mu^f_\kappa$ in FKG sense, provided the probabilities $p_{\{x,y\}}$ in μ^f_κ are given by (1.1) with β replaced by $\beta' \ge \kappa\beta$. Hence the above result extends to μ^f_κ when $\beta > \kappa$. Since $\mu^f_\kappa \le \mu^w_\kappa$, the same holds as well for μ^w_κ.

Remark 2 Theorem 1.1 should indeed extend exactly to the FK random cluster model with $\kappa > 1$. The authors believe that using an algebraic implementation of the multiscale analysis developed in the present work, one should be able to obtain this extension. Nevertheless, for the moment we do not have a full proof ([15]).

[1] That is $\mu \le \mu'$ if $\mu(g) \le \mu'(g)$ for any g continuous and increasing.

Some Related Problems The type of questions treated here has various sources of interest and we mention only a couple of them, which have to do with our own motivations. Consider the following physical problem: take the one-dimensional Ising model with pair interactions, the couplings decaying as the inverse of square of the distance between vertices, at inverse temperature $\beta > 1$; this is the model studied by Fröhlich and Spencer ([8]), for which a phase transition was established. Take now the finite box $[-L, L]$ and assume the Dobrushin boundary conditions, i.e. all spins in $(-\infty, -L]$ will be taken as $+1$, and all spins in $[L, +\infty)$ will be taken as -1. What can we say about the behaviour of this model when $L \to \infty$? Is there any sort of well defined *interface*? This might require a direct analysis in terms of the spin system, but it leads to a more general question for the FK model, regarding the behavior of connected components of each boundary conditioned not to touch each other. (Recall that by a random coloring of the clusters, the FK model gives origin to a spin system which interpolates the independent percolation model ($\kappa = 1$), the Ising model ($\kappa = 2$) and the q–states Potts model ($\kappa = q > 2$, integer) at inverse temperature β and interaction $J_{\{x,y\}} = \beta^{-1} \log(\frac{1}{1-p_{\{x,y\}}})$, the representation being possible for some (but not all) boundary conditions. For details see [2, 6]). Though we still do not fully understand this problem which remains unsolved, our results might shed some light on it. In [4], the authors obtain a more precise description for very low temperatures, using cluster expansion techniques.

An interesting corollary of Theorem 1.1 is as follows. Consider the Ising model (with ± 1-valued spins) on $\mathbb{Z}_+$, with interaction $J_{\{x,y\}} = |x-y|^{-2}$ if $|x-y| \geq 2$ and $J_{\{x,x+1\}} = J$ at inverse temperature β. Let $m_L^{0,+}(\beta)$ denote the average spin at the origin, with "one-sided" $(+)$ boundary conditions in $[L, \infty)$. By the above mentioned FK representation (see e.g. [2, 5, 11]), we have

$$m_L^{0,+}(\beta) = \mu^{w_r}_{2,[0,L]}(0 \leftrightarrow +\infty),$$

where $\mu^{w_r}_{2,[0,L]}$ stands for the random cluster measure on $\{0,1\}^{\mathbb{E}(\mathbb{Z}_+)}$ with $\kappa = 2$ and all the edges $\{x, y\}$ with $x \geq L$ and $y \geq L$ being open (wired on the right). Together with Remark 1 following Theorem 1.1, this yields the following

Corollary 1.2 *For any $\beta > 2$, there exist $0 < p_0 < 1$ such that, if $p > p_0$, then*

$$\lim_{L\to\infty} m_L^{0,+}(\beta) \geq \mu^{f}_{2,\mathbb{Z}_+}(0 \rightsquigarrow \infty) \geq \nu(0 \rightsquigarrow \infty) \geq 1 - \epsilon$$

holds for $\epsilon = \epsilon(J) \searrow 0$ as $J \nearrow \infty$. Consequently, there exists a phase transition when the thermodynamical limit on $\mathbb{Z}_+$ is taken with $+$ boundary conditions on the right side.

Remark In the above corollary there is a little change of notation with respect to the previously mentioned FK measure: the measure $\mu^{f}_{2,\mathbb{Z}_+}$ is considered here on $\{0,1\}^{\mathbb{E}(\mathbb{Z}_+)}$.

It is also interesting to compare the result on oriented percolation and the previous corollary with the somehow similar question on the multiplicity of Gibbs states for Markov chains with infinite connections, where orientation appears naturally through the time direction. Recently Johansson and Öberg [12] showed that if g is a regular specification and

$$\mathrm{var}_k(g) = \sup\{\|g(\sigma) - g(\sigma')\|_1 : \sigma_i = \sigma'_i, i = 1, \ldots, k\},$$

then g admits a unique Gibbs measure whenever the sequence $\{\mathrm{var}_k(g)\}_{k=1}^{+\infty}$ is in ℓ^2. This tells, in particular, that there are no multiple limiting measures for chains with connections decaying as r^{-2}, as in Example 1 in [12]. This contrasts with the two-sided Ising models and, as our Theorem says, with percolation models. The understanding of Markov chains with infinite connections in the non-uniqueness regime is still very poor, and it is known as a notoriously difficult problem. There is strong evidence (see [3]) that multi-scale analysis techniques analogous to those developed in this work could be turned into a robust tool to study this question.

Heuristics of the Proof The proof relies on Fröhlich-Spencer multi-scale analysis ideas [7, 8], and we use the version developed in [13, 14]. In the next few paragraphs we outline the scheme of the proof, and comment on some key ideas, avoiding most of consuming technical points. Our goal here is only to give a very schematic and approximate picture, postponing precise formulations (which tend to be quite involved) to later in the text.

The Goal We look for an event of positive probability, whose occurrence implies not only the existence of an infinite open component, but also guarantees the presence of an oriented infinite open path. Essentially, we will construct such an event, and show that it has positive probability. Our key estimate will be: if $\beta > 1$, we can find $\delta > 0$, $\delta' < 1$ and p sufficiently close to 1, so that

$$\begin{aligned}\nu(\exists \text{ open path } \pi = (x_1, e_1, \ldots, x_n)\colon x_1 \le -L + L^{\delta'},\ x_n \ge L - L^{\delta'}, 0 < x_i - x_{i-1} \le L^{\delta'},\\ \forall i\,) \ge 1 - 2L^{-\delta} \qquad (1.4)\end{aligned}$$

for $L = l_k$ as defined below and any $k \ge 1$, l_1 being sufficiently large, and where ν stands for the product measure defined before. We will have little control on how close to one p has to be (or, equivalently, on how large we need l_1).

Scales We choose super-exponentially fast growing scales. Given $1 < \alpha < 2$, $l_0 = 1$ and l_1 an integer sufficiently large, let

$$l_k = \lfloor l_{k-1}^{\alpha-1} \rfloor l_{k-1}, \quad k = 2, 3, \ldots, \qquad (1.5)$$

where as usual $\lfloor z \rfloor = \max\{n \in \mathbb{N}\colon n \le z\}$. We will use the so-called dynamical blocking argument, where the size and location of blocks[2] will be defined along the procedure and will depend on the configuration at lower scales. Still, the length of each block $I^{(k)}$ of the k-th level (called k-block) will be of order l_k. More precisely, we shall see that $l_k - 2l_k^{\alpha'/\alpha} - 6l_{k-1} \le |I^{(k)}| \le 3l_k + 6l_{k-1}$, for suitable $1 < \alpha' < \alpha$. (In particular, $|I^{(k)}| \ll l_{k+1}^{\alpha'/\alpha} \ll l_{k+1}$, if $k \ge 1$ and l_1 is large.)

Defected and Good Blocks Further we will use the following recursive definition of "defected" block. Fix $1 < \alpha' < \alpha$ to be specified later.

(1) We say that the 0-block $[i, i+1]$ is *defected* if the corresponding nearest neighbor edge $\{i, i+1\}$ is closed; otherwise the 0-block is said to be *good* and the open nearest neighbor path from i to $i+1$ is called a 0–*pedestal*;
(2) For $k \ge 1$, a k-block $I^{(k)} = [s, s']$ is *defected* if either it contains two or more defected $(k-1)$-blocks, or it contains only one defected $(k-1)$-block $[i, i']$ but there is no open edge $\{a, a'\}$ of length at most $l_k^{\alpha'/\alpha}$, with $a \le i$, $i' \le a'$, $a \in \Upsilon$, $a' \in \Upsilon'$, for some $(k-1)$-pedestals Υ, Υ' contained in $I^{(k)}$. Otherwise $I^{(k)}$ is called *good*.

[2] Successive blocks share an end-vertex.

Thus, if a k-block $[s, s']$ is good, then it contains an oriented open path going from s to s': in the case it has no defected $(k-1)$-blocks, this path can be obtained by concatenating $(k-1)$-pedestals of the good $(k-1)$-blocks which constitute the given k-block; if it has a (single) defected $(k-1)$-block, a similar concatenation yields an oriented open path going from s to a, which is followed by an open edge $\{a, a'\}$, and then followed by another concatenation of $(k-1)$-pedestals of good $(k-1)$-blocks, from a' to s'. In both cases, such path from s to s' will be called *k–pedestal*, and denoted by Υ. The part of the cluster between a and a' is again disregarded in the future construction since we have little control on oriented connectivity in this segment. The condition $a' - a \le l_k^{\alpha'/\alpha}$ will be crucial to guarantee that pedestals are quite dense sets (within the corresponding good blocks), used to push the construction to higher levels. Some care is needed when treating defects close to the boundary, which we have disregarded here.

Strategy Being "defected" doesn't necessarily imply that there is no oriented open path connecting the endpoints of the block. Nevertheless, in order to avoid substantial technical difficulties, we will follow two rules that simplify our construction:

(a) once a block is defected, we will assume the worst possible situation, namely it will be considered as if all edges within this block were closed.
(b) once we have at least two defected $(k-1)$-blocks within a k-block I, we will not try to find connections within the k-block to fix its connectivity, but rather will "push the problem to the next level", and try to "jump over" this troubled block I by a longer edge of length at most $l_{k+1}^{\alpha'/\alpha}$, which starts at the pedestal of some good k-block to the left of I, and ends similarly on the right of I.

Estimates The scale l_1 will be taken large enough, to be determined later depending on the parameter $\beta > 1$ and the auxiliary parameters $\delta > 0$, $1 < \alpha' < \alpha < 2$, to be chosen at the end of Sect. 2 (see (2.17)–(2.20)). Once l_1 is chosen, we shall take p so that:

$$p \ge \left(1 + \frac{(\ln 2)^5}{128}\, l_1^{-\delta-1}\right)^{-1}. \tag{1.6}$$

For $k \ge 2$, let $I^{(k)}$ be a k-block of length[3] l_k, which consists of $N_k = l_k/l_{k-1} = \lfloor l_{k-1}^{\alpha-1} \rfloor$ blocks of level $(k-1)$, of length l_{k-1}, and written as $\{I_j^{(k-1)}\}_{j=1}^{N_k}$. Assume that we have the following estimate

$$\nu(I_j^{(k-1)} \text{ is defected}) \le l_{k-1}^{-\delta}, \quad 1 \le j \le N_k.$$

Under the above assumptions, and if δ is chosen to satisfy (2.18), we see that

$$\nu\left(\exists 1 \le i < j \le N_k\colon\, I_i^{(k-1)}, I_j^{(k-1)} \text{ are both defected}\right) \le \frac{1}{2}\, l_k^{-\delta}. \tag{1.7}$$

When the defected $I_i^{(k-1)}$ is unique, we assume for the moment that it stays at distance larger than $l_k^{\alpha'/\alpha}$ from the boundary of $I^{(k)}$. (Otherwise a sequence of local adjustments of blocks will be needed, as we shall see in Sect. 2. The left- and right-most extremal blocks in our volume are treated differently.) In this case let a and a' be the end-vertices of the unique defected block $I_i^{(k-1)}$. By our construction, there exists an oriented path starting from the

[3] This is not exact in general, but holds approximately, cf. (2.7).

left boundary of $I^{(k)}$ and ending at the vertex a and another open oriented path starting from vertex a' and going to the right boundary of $I^{(k)}$. Both these paths are obtained by concatenating pedestals of all good $(k-1)$-blocks on the left side of the defected block $I_i^{(k-1)}$ and, respectively, on the right side. We denote these new left and right pedestals by Υ and Υ', respectively. Given that $I^{(k)}$ has a unique defected $I_i^{(k-1)} = [a, a']$, and given the pedestals Υ and Υ', one has the following upper bound for the conditional ν–probability of not finding an open edge $\{x, y\}$ with $x \le a, a' \le y,\ x \in \Upsilon,\ y \in \Upsilon'$ and $y - x \le l_k^{\alpha'/\alpha}$:

$$\prod_{\substack{x,y: x\le a < a' \le y, \\ y-x<l_k^{\alpha'/\alpha} \\ x\in\Upsilon,\, y\in\Upsilon'}} q_{\{x,y\}} = \exp\left\{-\sum_{\substack{x,y: x\le a < a' \le y, \\ y-x<l_k^{\alpha'/\alpha} \\ x\in\Upsilon,\, y\in\Upsilon'}} \frac{\beta}{|x-y|^2}\right\} \le l_{k-1}^{-\beta(1-\eta)(\alpha'-1)}, \tag{1.8}$$

where $\eta = \eta(\alpha, \alpha', l_1) > 0$, and can be taken arbitrarily small if $l_1 \to \infty$.

The precise statement and proof of the above estimate will be given in Lemma 2.1. It requires some work, and in order to obtain it for suitable $\eta = \eta(\alpha, \alpha', l_1) > 0$ which can be taken arbitrarily small if $l_1 \to \infty$ we will need to use certain geometric properties of pedestals Υ and Υ', which propagate inductively from each level into the next one. Namely, the pedestals are relatively dense sets (see (2.8) in Sect. 2) as the construction will show. Using the above estimate, writing

$$\begin{aligned}
&\left\{I^{(k)} \text{ has a unique defected } (k-1)\text{-block } [a,a'] \text{ and remains defected}\right\} \\
&\quad = \left\{I^{(k)} \text{ has unique defected } (k-1)\text{-block } [a,a']\right\} \\
&\qquad \cap \left\{ \text{there is no open edge } \{x,y\} \text{ with } x \le a, a' \le y,\ x \in \Upsilon,\ y \in \Upsilon' \text{ and } y - x \le l_k^{\alpha'/\alpha}\right\},
\end{aligned}$$

and since these events depend on disjoint sets of edges, we easily get:

$$\nu\left(I^{(k)} \text{ has a unique defected } I_i^{(k-1)} \text{ and remains defected}\right) \le l_{k-1}^{\alpha-1-\delta} l_{k-1}^{-\beta(1-\eta)(\alpha'-1)} \le \frac{1}{2} l_k^{-\delta}, \tag{1.9}$$

provided

$$\beta(1-\eta)(\alpha'-1) > (\delta+1)(\alpha-1). \tag{1.10}$$

Since $\beta > 1$ and $\eta = \eta(\alpha, \alpha', l_1)$ can be taken very small provided l_1 is large, it will suffice to suitably fix the parameters α and α' (α' close enough to α). This is done at the end of Sect. 2.

Difficulties To carry on this scheme we have to go through several "unpleasant" and rather involved points. The use of a dynamical blocking argument, with the blocks of a given level depending not only on the size and location of lower level blocks, but also on their "status" (defected or good), requires a rather tight bookkeeping. This is expressed through what we call "itineraries".

Once this is achieved, all necessary estimates follow along the scheme of [8, 13].

In the next section we define the blocks and describe the dynamic renormalization procedure, proving Theorem 1.1.

2 Spatial Blocks (Dynamic Renormalization)

Notation For $L \in \mathbb{N}$, assumed to be large, the construction will involve the configuration ω restricted to the set of edges with both end-vertices in $[-L, L]$, where $[a,b] = [a,b] \cap \mathbb{Z}$ throughout.[4] We write Ω_L as a shorthand for $\Omega_{[-L,L]}$. Scales $\{l_k\}_{k\in\mathbb{N}}$ are defined in the following way: $l_0 = 1$, given $\beta > 1$ we shall take auxiliary parameters $\delta > 0$, $1 < \alpha' < \alpha < 2$ chosen according to (2.17)–(2.20), l_1 will be a suitably large integer and the parameter $p < 1$ will be taken sufficiently close to 1, depending on l_1. Then we let l_k be given by (1.5).

Further we denote $x_j^{(k)} = jl_k,\ j \in \mathbb{Z}$.
For the proof of Theorem 1.1 we may assume that $L = l_M$, for some $M \in \mathbb{N}$.
Throughout the text $\mathbb{I}_A$ stands for the indicator function of an event A, i.e. $\mathbb{I}_A(\omega) = 1$ or 0 according to $\omega \in A$ or not.

Decomposition of Events. Level 0 We set $I_i^{(0)} = [i, i+1]$. They are called 0-blocks, and for i such that $I_i^{(0)} \subset [-L, L]$ we define the events:

$$G(I_i^{(0)}) = \{\omega : \omega_{\{i,i+1\}} = 1\}, \qquad B(I_i^{(0)}) = \{\omega : \omega_{\{i,i+1\}} = 0\}.$$

$I_i^{(0)}$ is said to be defected when $B(I_i^{(0)})$ occurs; otherwise it is said to be a good 0-block.

Level 1 Consider the intervals $\widetilde{I}_j^{(1)} = [jl_1, (j+1)l_1]$ and for each j such that $\widetilde{I}_j^{(1)} \subset [-L, L]$ we define the following partition of Ω_L:

$$\begin{aligned}
G(\widetilde{I}_j^{(1)}) &= \bigcap_{i=jl_1}^{(j+1)l_1-1} G(I_i^{(0)}),\\
H_i(\widetilde{I}_j^{(1)}) &= B(I_i^{(0)}) \cap \bigcap_{\substack{s=jl_1\\ s\neq i}}^{(j+1)l_1-1} G(I_s^{(0)}) \quad \text{for } i \in [jl_1, (j+1)l_1 - 1],\\
H(\widetilde{I}_j^{(1)}) &= \bigcup_{i=jl_1}^{(j+1)l_1-1} H_i(\widetilde{I}_j^{(1)}),\\
B(\widetilde{I}_j^{(1)}) &= \left(G(\widetilde{I}_j^{(1)}) \cup H(\widetilde{I}_j^{(1)})\right)^c,
\end{aligned} \tag{2.1}$$

where G stands for *good*, H for *hopeful* and B for *bad*, and accordingly, $\widetilde{I}_j^{(1)}$ is said to be good (for given ω) if it contains no defected 0-blocks, "hopeful" if it contains only one defected 0-block, and is said to be "bad" otherwise. When $H_i(\widetilde{I}_j^{(1)})$ occurs, $I_i^{(0)}$ is called the defected 0-block in $\widetilde{I}_j^{(1)}$.

Adjustment Given ω, we first consider the set of all j's such that $\omega \in H_{i_j}(\widetilde{I}_j^{(1)}) \subset H(\widetilde{I}_j^{(1)})$ and such that the index i_j of the (unique) defected block $I_{i_j}^{(0)} \subset \widetilde{I}_j^{(1)}$ verifies $jl_1 \le i_j \le jl_1 + \lfloor l_1^{\alpha'/\alpha} \rfloor - 1$ (resp. $(j+1)l_1 - \lfloor l_1^{\alpha'/\alpha} \rfloor \le i_j \le (j+1)l_1 - 1$).

[4] Except in the proof of Lemma 2.1.

If this set is empty in both cases, we set $I_j^{(1)} = \widetilde{I}_j^{(1)}$ for all j's, and say that $G(I_j^{(1)})$, $H(I_j^{(1)})$, $B(I_j^{(1)})$ occurs, according to the occurrence of the corresponding $G(\widetilde{I}_j^{(1)})$, $H(\widetilde{I}_j^{(1)})$, $B(\widetilde{I}_j^{(1)})$.

If this set is not empty, we take arbitrarily one of such indices j; if $\widetilde{I}_j^{(1)}$ is not the interval which contains $-L$ (resp. L), to be treated in case 3) below, we check if $\widetilde{I}_{j-1}^{(1)}$ (resp. $\widetilde{I}_{j+1}^{(1)}$) has a defected 0-block in the sub-interval $[jl_1 - 2\lfloor l_1^{\alpha'/\alpha}\rfloor, jl_1]$ (resp. $[(j+1)l_1, (j+1)l_1 + 2\lfloor l_1^{\alpha'/\alpha}\rfloor - 1]$).

(1) If yes, then we consider a new interval $I_{j-1}^{(1)} = \widetilde{I}_{j-1}^{(1)} \cup \widetilde{I}_j^{(1)}$ (resp. $I_j^{(1)} = \widetilde{I}_j^{(1)} \cup \widetilde{I}_{j+1}^{(1)}$) and say that the event $B(I_{j-1}^{(1)})$ (resp. $B(I_j^{(1)})$) occurs. (This is motivated by the fact that for the chosen ω the new interval will contain at least two defected 0-blocks.)
(2) If not, then we consider two new intervals $I_{j-1}^{(1)} = [(j-1)l_1, jl_1 - \lfloor l_1^{\alpha'/\alpha}\rfloor]$ and $I_j^{(1)} = [jl_1 - \lfloor l_1^{\alpha'/\alpha}\rfloor, (j+1)l_1]$ (resp. $I_j^{(1)} = [jl_1, (j+1)l_1 + \lfloor l_1^{\alpha'/\alpha}\rfloor]$ and $I_{j+1}^{(1)} = [(j+1)l_1 + \lfloor l_1^{\alpha'/\alpha}\rfloor, (j+2)l_1]$). We say that $H(I_j^{(1)})$ occurs, and that $G(I_{j-1}^{(1)})$, $H(I_{j-1}^{(1)})$, $B(I_{j-1}^{(1)})$ occurs according to the occurrence of the corresponding event $G(\widetilde{I}_{j-1}^{(1)})$, $H(\widetilde{I}_{j-1}^{(1)})$, $B(\widetilde{I}_{j-1}^{(1)})$ (resp. we say that $H(I_j^{(1)})$ occurs, and that $G(I_{j+1}^{(1)})$, $H(I_{j+1}^{(1)})$, $B(I_{j+1}^{(1)})$ occurs according to the occurrence of the corresponding event $G(\widetilde{I}_{j+1}^{(1)})$, $H(\widetilde{I}_{j+1}^{(1)})$ $B(\widetilde{I}_{j+1}^{(1)})$). In this case the adjustment moves the boundary "away" from the unique defected block in $I_j^{(1)}$, but doesn't change the number of the defected 0-blocks in the adjusted intervals.
(3) If the interval $\widetilde{I}_j^{(1)}$ under consideration is the leftmost (resp. the rightmost) interval in $[-L, L]$, and the defect stays within distance less than $\lfloor l_1^{\alpha'/\alpha}\rfloor$ from $-L$ (resp. L), we still set $I_j^{(1)} = \widetilde{I}_j^{(1)}$ and say that $G(I_j^{(1)})$ occurs.
(4) We set $I_j^{(1)} = \widetilde{I}_j^{(1)}$ if $\widetilde{I}_j^{(1)}$ was not involved in the previous adjustment, and say that $G(I_j^{(1)})$, $H(I_j^{(1)})$, $B(I_j^{(1)})$ occurs if, accordingly, $G(\widetilde{I}_j^{(1)})$, $H(\widetilde{I}_j^{(1)})$, $B(\widetilde{I}_j^{(1)})$ occurs.

To conclude this step, we re-numerate the intervals from left to right as $I_j^{(1)}$ $j = 1, \ldots$. If we are still left with intervals $I_j^{(1)}$ for which $H(I_j^{(1)})$ occurs and its defected 0-block stays within distance $\lfloor l_1^{\alpha'/\alpha}\rfloor$ from the boundary of $I_j^{(1)}$, we repeat the above procedure to the intervals already adjusted in the previous step. After finitely many steps of such adjustment procedure there are left no intervals $I_j^{(1)}$ for which the event $H(I_j^{(1)})$ occurs and its defected 0-block stays within distance $\lfloor l_1^{\alpha'/\alpha}\rfloor$ from the boundary, and the adjustment procedure is then stopped. (Of course, due to item 3, the left- or rightmost intervals can stay with a unique defect, if this is close enough to $-L$ or L respectively.)

Remark Notice that the adjustment procedure is well defined, i.e. the final partition does not depend on the order in which we do adjustments and in which order we pick the intervals that still need to be adjusted (in case we have more than one). It also has a *locality* property, i.e. the final modification of each initial interval $\widetilde{I}_j^{(1)}$ depends on the values of the configuration in the nearest neighbor and, at most, in the next nearest neighbor intervals only.

Once the adjustment is completed, the obtained intervals, always re-numerated from left to right as $I_j^{(1)}$, $j = 1, \ldots$, are called 1-blocks. Notice that $l_1 - 2\lfloor l_1^{\alpha'/\alpha}\rfloor \le |I_j^{(1)}| \le 3l_1$, and $\bigcup_j I_j^{(1)} = [-L, L]$.

In other words, the restriction of ω to nearest neighbor edges of $[-L, L]$ determines through the above procedure a random "partition" $I^{(1)}(\omega) \equiv \{I_j^{(1)}(\omega)\}_j$ of the interval

$[-L, L]$ into 1-blocks, with the property that any two adjacent blocks share an end-vertex. This is the final state of the "adjustment" procedure. Values of ω on the nearest neighbor edges in $[-L, L]$ also determine where the defected 0-blocks are located within each 1-block, and we denote by $D_j^{(1)}(\omega)$ the set of indices of the defected 0-blocks within $I_j^{(1)}(\omega)$, and $D^{(1)}(\omega) \equiv \{D_j^{(1)}(\omega)\}_j$. The random object $J_L^{(1)} := \{I_j^{(1)}, D_j^{(1)}\}$ is called itinerary at level 1 or 1-itinerary.

1-Pedestals Given the 1-itinerary $J_L^{(1)}$, we shall attribute to each random block $I_j^{(1)}$ a state G or B. We first consider the case that $I_j^{(1)}$ is not the leftmost (i.e. $j \neq 1$) nor the rightmost 1-block, to be treated at the end. When $D_j^{(1)} = \emptyset$, so that all nearest neighbor edges are open, we say that $I_j^{(1)}$ is in state G, and we define the pedestal $\Upsilon(I_j^{(1)}) = I_j^{(1)}$. When $|D_j^{(1)}| = 1$, the set of vertices $x \in I_j^{(1)}$ to the left (resp. right) of the defected 0-block in $I_j^{(1)}$ will be called *left 1-pedestal of* $I_j^{(1)}$ (resp. *right 1-pedestal*) and denoted by $\Upsilon_{\mathcal{L}}(I_j^{(1)})$ (resp. $\Upsilon_{\mathcal{R}}(I_j^{(1)})$). The vertices in each of these 1-pedestals are connected by open nearest neighbor edges. In this situation we say that $I_j^{(1)}$ is in state G when the following event occurs:

$$[\omega : \exists x \in \Upsilon_{\mathcal{L}}(I_j^{(1)}(\omega)), y \in \Upsilon_{\mathcal{R}}(I_j^{(1)}(\omega)),\ 1 < y - x \leq \lfloor l_1^{\alpha'/\alpha} \rfloor : \omega_{\{x,y\}} = 1], \tag{2.2}$$

and otherwise we say that $I_j^{(1)}$ is in state B. Similarly, if $|D_j^{(1)}| > 1$ the block $I_j^{(1)}$ is in state B.

For the leftmost (rightmost) 1-block, there is some little difference: In the case $|D_j^{(1)}| = 1$ and if the unique defected 0-block stays within distance $\lfloor l_1^{\alpha'/\alpha} \rfloor$ from $-L$ (L), the block is said to be in state G, and the pedestal is defined as the previously defined right 1-pedestal (left 1-pedestal, resp.), $\Upsilon(I_1^{(1)}) = \Upsilon_{\mathcal{R}}(I_1^{(1)})$ ($\Upsilon(I_j^{(1)}) = \Upsilon_{\mathcal{L}}(I_j^{(1)})$, resp.). Except for this, the definition goes as with the other blocks.

With a little abuse of notation we use again the symbols $G(I_j^{(1)})$ and $B(I_j^{(1)})$ to denote that $I_j^{(1)}$ is in state G and B respectively. We say that $I_j^{(1)}(\omega)$ is *defected* if and only if it is in state B.

In (2.2), if the pair (x, y) such that $x \in \Upsilon_{\mathcal{L}}(I_j^{(1)})$, $y \in \Upsilon_{\mathcal{R}}(I_j^{(1)})$, $y - x \leq \lfloor l_1^{\alpha'/\alpha} \rfloor$, $\omega_{\{x,y\}} = 1$ is not unique, we choose one in arbitrary way, and, once the pair (x, y) is chosen, the interval $[x + 1, y - 1]$ will be called defected part of $I_j^{(1)}$, and denoted by $\mathcal{D}(I_j^{(1)})$. In this case we define $\Upsilon(I_j^{(1)}) = (\Upsilon_{\mathcal{L}}(I_j^{(1)}) \cup \Upsilon_{\mathcal{R}}(I_j^{(1)})) \setminus \mathcal{D}(I_j^{(1)})$.

In particular, a 1-pedestal $\Upsilon(I_j^{(1)})$ is given by the vertices of an open oriented path with all edges, except possibly one, being nearest neighbor, and this larger edge has length at most $\lfloor l_1^{\alpha'/\alpha} \rfloor$. For each 1-block, except possibly the two which contain the extremes $-L$ or L, the pedestal connects left and right endpoints of the interval. In the leftmost (rightmost) case, it is allowed for the 1-pedestal to start (end) at a vertex within distance $\lfloor l_1^{\alpha'/\alpha} \rfloor + 1$ of $-L$ (L respectively).

Level k Let $2 \leq k \leq M$. Assume to have completed the step $(k - 1)$ of the recursion. In particular, for each $\omega \in \Omega_L$ and any $r = 1, \ldots, k - 1$ the following objects are defined:

- the collection of r-blocks $I^{(r)}(\omega) = \{I_j^{(r)}(\omega)\}_j$, such that $\bigcup_j I_j^{(r)}(\omega) = [-L, L]$, and any two adjacent intervals share exactly an endpoint. Moreover, the uniform bound holds:

$$l_r - (2\lfloor l_r^{\alpha'/\alpha} \rfloor + 6l_{r-1}) < |I_j^{(r)}(\omega)| \leq 3l_r + 6l_{r-1}, \tag{2.3}$$

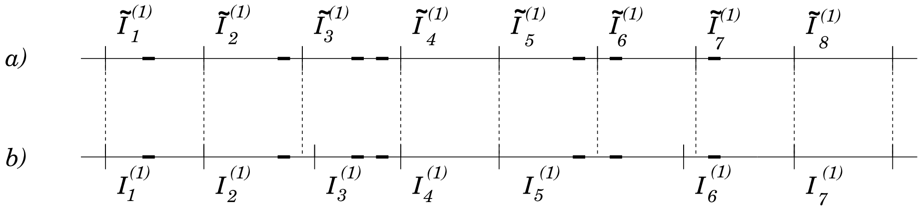

Fig. 1 Adjustments: part (**a**) shows the deterministic 1-blocks $\widetilde{I}_j^{(1)}$, bold-face segments show location of the defects. Part (**b**) shows how these blocks were adjusted. $\widetilde{I}_5^{(1)}$ and $\widetilde{I}_6^{(1)}$, merge into a single 1-block $I_5^{(1)}$

- each of the $I_j^{(r)}(\omega)$ can be in two possible states G or B:
  If $I_j^{(r)}(\omega)$ is in state G and it is not the leftmost or the rightmost interval of the partition, then ω has an r-pedestal $\Upsilon(I_j^{(r)})$ given by vertices of an open oriented path from the left to the right boundary of $I_j^{(r)}(\omega)$. If $I_j^{(r)}(\omega)$ is the leftmost (resp. the rightmost) interval, an r-pedestal $\Upsilon(I_j^{(r)})$ is given by vertices of an open oriented path which starts from some vertex $x \in [-L, -L + 2\lfloor l_r^{\alpha'/\alpha} \rfloor]$ and ends at the right boundary of $I_j^{(r)}(\omega)$ (resp. starts from the left boundary of $I_j^{(r)}(\omega)$ and ends at some vertex $x \in [L - 2\lfloor l_r^{\alpha'/\alpha} \rfloor, L]$). ($l_1$ being large, we may assume that the length of an $(r-1)$-block is always bounded above by $\lfloor l_r^{\alpha'/\alpha} \rfloor$, according to (2.3) for r replaced by $r-1$.)
- the collection $D^{(r)}(\omega) = \{D_j^{(r)}(\omega)\}_j$, where $D_j^{(r)}(\omega)$ is the set of labels of the defected $(r-1)$-blocks which are contained in $I_j^{(r)}(\omega)$.

For ω fixed, the sequence of pairs

$$J_L^{(k-1)}(\omega) = \{(I^{(1)}(\omega), D^{(1)}(\omega)), \dots, (I^{(k-1)}(\omega), D^{(k-1)}(\omega))\},$$

is called $(k-1)$-*itinerary*, and $(I^{(r)}, D^{(r)})$, is called the r-th step of the itinerary, for $1 \le r \le k-1$. We shall now see how to define the k-blocks and the continuation to a k-itinerary. When $k = M$ we will end up with only one or two intervals.

Construction of k-blocks For any ω and for each $z \in [-L, L]$ we set $j_z^k = \min\{j : z \in I_j^{(k-1)}\}$, $\hat{j}_i^k = j_{x_i^{(k)}}^k$, cf. notation at the beginning of this section, $i = -l_M/l_k, \dots, l_M/l_k - 1$, and define the intervals:

$$\widetilde{I}_i^{(k)} = \bigcup_{s=\hat{j}_i^k+1}^{\hat{j}_{i+1}^k} I_s^{(k-1)} =: [a_i^{(k)}, a_{i+1}^{(k)}]$$

as well as the following partition of Ω_L:

$$G(\widetilde{I}_i^{(k)}) = \bigcap_{s=\hat{j}_i^k+1}^{\hat{j}_{i+1}^k} G(I_s^{(k-1)}),$$

$$H_s(\widetilde{I}_i^{(k)}) = B(I_s^{(k-1)}) \cap \bigcap_{u=\hat{j}_i^k+1, u \neq s}^{\hat{j}_{i+1}^k} G(I_u^{(k-1)}), \tag{2.4}$$

$$H(\widetilde{I}_i^{(k)}) = \bigcup_{s=\hat{j}_i^k+1}^{\hat{j}_{i+1}^k} H_s(\widetilde{I}_i^{(k)}),$$

$$B(\widetilde{I}_i^{(k)}) = \Omega_L \setminus \left(G(\widetilde{I}_i^{(k)}) \cup H(\widetilde{I}_i^{(k)}) \right).$$

Adjustment Given $\omega \in \Omega_L$, consider all i for which $H_s(\widetilde{I}_i^{(k)})$ occurs for s such that the distance of the defected $(k-1)$-block $I_s^{(k-1)} \subset \widetilde{I}_i^{(k)}$ to the left endpoint $a_i^{(k)}$ (right endpoint $a_{i+1}^{(k)}$, resp.) is less than $\lfloor l_k^{\alpha'/\alpha} \rfloor$. If this set is non-empty take arbitrarily any such $\widetilde{I}_i^{(k)}$.

When the selected $\widetilde{I}_i^{(k)}$ is the leftmost (resp. the rightmost) interval in $[-L, L]$, and the defect stays at distance less than $\lfloor l_k^{\alpha'/\alpha} \rfloor$ from $-L$ (resp. L), we set $I_i^{(k)} = \widetilde{I}_i^{(k)}$, and say that $G(I_i^{(k)})$ occurs (or that $I_i^{(k)}$ is in G state for this ω). Otherwise, we then check if $\widetilde{I}_{i-1}^{(k)}$ (resp. $\widetilde{I}_{i+1}^{(k)}$) has a defected block $I_r^{(k-1)}$ at distance at most $3\lfloor l_k^{\alpha'/\alpha} \rfloor$ from $a_i^{(k)}$ (resp. from $a_{i+1}^{(k)}$), and

(1) If yes, then we consider a new interval $I_{i-1}^{(k)} = \widetilde{I}_{i-1}^{(k)} \cup \widetilde{I}_i^{(k)}$ (respectively $I_i^{(k)} = \widetilde{I}_i^{(k)} \cup \widetilde{I}_{i+1}^{(k)}$) and say that $B(I_{i-1}^{(k)})$ (resp. $B(I_i^{(k)})$) occurs, or that the corresponding interval is in state B;
(2) If not, then we consider two new intervals:

$$I_{i-1}^{(k)} = \bigcup_{s=\hat{j}_{i-1}^k+1}^{j^k_{a_i^{(k)} - l_k^{\alpha'/\alpha}} - 1} I_s^{(k-1)}, \qquad I_i^{(k)} = \bigcup_{s=j^k_{a_i^{(k)} - l_k^{\alpha'/\alpha}}}^{\hat{j}_{i+1}^k} I_s^{(k-1)} \tag{2.5}$$

$$\left(\text{respectively,} \quad I_i^{(k)} = \bigcup_{s=\hat{j}_i^k+1}^{j^k_{a_{i+1}^{(k)} + l_k^{\alpha'/\alpha}}} I_s^{(k-1)}, \qquad I_{i+1}^{(k)} = \bigcup_{s=j^k_{a_{i+1}^{(k)} + l_k^{\alpha'/\alpha}} + 1}^{\hat{j}_{i+2}^k} I_s^{(k-1)} \right). \tag{2.6}$$

In the situation of (2.5) we say that $H(I_i^{(k)})$ occurs, and say that $G(I_{i-1}^{(k)})$, $H(I_{i-1}^{(k)})$, $B(I_{i-1}^{(k)})$ occurs according to the occurrence of the corresponding $G(\widetilde{I}_{i-1}^{(k)})$, $H(\widetilde{I}_{i-1}^{(k)})$, $B(\widetilde{I}_{i-1}^{(k)})$ (resp. in the situation of (2.6) we say that $H(I_{i-1}^{(k)})$ occurs, and say that $G(I_{i+1}^{(k)})$, $H(I_{i+1}^{(k)})$, $B(I_{i+1}^{(k)})$ occurs according to the occurrence of $G(\widetilde{I}_{i+1}^{(k)})$, $H(\widetilde{I}_{i+1}^{(k)})$, $B(\widetilde{I}_{i+1}^{(k)})$).

Finally we set $I_i^{(k)} = \widetilde{I}_i^{(k)}$ if $\widetilde{I}_i^{(k)}$ was not involved in the adjustment and say $G(I_i^{(k)})$, $H(I_i^{(k)})$, $B(I_i^{(k)})$ occurs if the corresponding $G(\widetilde{I}_i^{(k)})$, $H(\widetilde{I}_i^{(k)})$, $B(\widetilde{I}_i^{(k)})$ does occur.

To conclude this step, we re-numerate the intervals from left to right as $I_j^{(k)}$ $j = 1, \ldots$. If after this step we are still left with intervals $I_i^{(k)}$ for which $H(I_i^{(k)})$ occurs and its defected interval $I_s^{(k-1)}$ stays within distance $\lfloor l_k^{\alpha'/\alpha} \rfloor$ from one of the endpoints of $I_i^{(k)}$, then we repeat the above procedure. After finitely many steps of this adjustment procedure all $I_i^{(k)}$ for which $H(I_i^{(k)})$ occurs have their defected $(k-1)$-block at distance larger than $\lfloor l_k^{\alpha'/\alpha} \rfloor$ from the boundary of $I_i^{(k)}$.

Once the adjustments are completed, the final intervals, always re-numerated from left to right as $I_j^{(k)}$, $j = 1, \ldots$, are called k-blocks. We then consider the collection $D^{(k)} = \{D_j^{(k)}\}_j$ where $D_j^{(k)}$ gives the labels of the defected $(k-1)$-blocks contained in $I_j^{(k)}$.

We can always write $I_j^{(k)} = \bigcup_{s_0(j)}^{s_1(j)} I_s^{(k-1)}$. It is easy to check that the procedure is well defined (measurable) and the validity of the following recursive estimate:

$$l_k - (2\lfloor l_k^{\alpha'/\alpha} \rfloor + 6l_{k-1}) < |I_j^{(k)}| \le 3l_k + 6l_{k-1}. \tag{2.7}$$

k-Pedestals Given the k-itinerary we shall associate to each k-block $I_j^{(k)}(\omega)$ a state G or B, and the blocks in state G will have a k-pedestal, to be defined below. When $|D_j^{(k)}(\omega)| \ge 2$, the block is said to be in state B, and it has no k-pedestal.

- When $D(I_j^{(k)}) = \emptyset$, all its sub-blocks $I_s^{(k-1)}$ are in state G. In this case we define $\Upsilon(I_j^{(k)}) = \bigcup_{s_0(j)}^{s_1(j)} \Upsilon(I_s^{(k-1)})$.
- If $D(I_j^{(k)}) = \{r\}$ and $I_j^{(k)}$ is not the leftmost (resp. rightmost) interval in $[-L, L]$, we define $\Upsilon_{\mathcal{L}}(I_j^{(k)}) = \bigcup_{s_0(j)}^{r-1} \Upsilon(I_s^{(k-1)})$ and $\Upsilon_{\mathcal{R}}(I_j^{(k)}) = \bigcup_{r+1}^{s_1(j)} \Upsilon(I_s^{(k-1)})$, called left and right pedestals[5] of $I_j^{(k)}$, and check if there exists $x \in \Upsilon_{\mathcal{L}}(I_j^{(k)})$ and $y \in \Upsilon_{\mathcal{R}}(I_j^{(k)})$ with $y - x \le \lfloor l_k^{\alpha'/\alpha} \rfloor$ such that $\omega_{\{x,y\}} = 1$:
  - If yes, we say that $I_j^{(k)}$ is in state G, and if the pair (x, y) with $x \in \Upsilon_{\mathcal{L}}(I_j^{(k)})$, $y \in \Upsilon_{\mathcal{R}}(I_j^{(k)})$, $y - x \le \lfloor l_k^{\alpha'/\alpha} \rfloor$, and $\omega_{\{x,y\}} = 1$ is not unique, we choose one in an arbitrary way, and, once (x, y) is chosen, denote $\mathcal{D}(I_j^{(k)}) = [x + 1, y - 1]$, and define

$$\Upsilon(I_j^{(k)}) = \left(\Upsilon_{\mathcal{L}}(I_j^{(k)}) \cup \Upsilon_{\mathcal{R}}(I_j^{(k)})\right) \setminus \mathcal{D}(I_j^{(k)}).$$

  - If such an open edge $\{x, y\}$ does not exist we say that $I_j^{(k)}$ is in B state.
- If $|D(I_j^{(k)})| = 1$ and $I_j^{(k)}$ is the leftmost (resp. rightmost) interval in $[-L, L]$ whose unique defected $(k-1)$-block $I_r^{(k-1)}$ stays within distance $\lfloor l_k^{\alpha'/\alpha} \rfloor$ from $-L$ (resp. L), then we say that $I_j^{(k)}$ is in state G and we define its k-pedestal as $\Upsilon(I_j^{(k)}) = \bigcup_{r+1}^{s_1(j)} \Upsilon(I_s^{(k-1)})$ (resp. $\Upsilon(I_j^{(k)}) = \bigcup_{s_0(j)}^{r-1} \Upsilon(I_s^{(k-1)})$).
- Finally, if $|D(I_j^{(k)})| = 1$ and $I_j^{(k)}$ is the leftmost (resp. rightmost) interval in $[-L, L]$, but its unique defected $(k-1)$-block $I_r^{(k-1)}$ does not stay within distance $\lfloor l_k^{\alpha'/\alpha} \rfloor$ from $-L$ (resp. L), then we use the same procedure as if $I_j^{(k)}$ were not an extremal k-block.

This completes the k-th step, associating with each itinerary $J^{(k-1)}$ its continuation with a random sequence of k-blocks $I^{(k)} = \{I_j^{(k)}\}_j$, re-numerated from left to right. Moreover, with each k-block we associate one of the states G or B.

Structure of Pedestals First we state a simple geometric property of pedestals, which will be used in estimating the conditional probability that a k-block $I_j^{(k)}$ is in state G, given that $|D_j^{(k)}| = 1$. Our goal is to show that there exists a positive constant $C \equiv C(\alpha, \alpha')$ such that if a k-block, $k \ge 1$, $I^{(k)} = [s, s']$ contains only one defected $(k-1)$-block, here denoted by

[5] From the occurrence of $G(I_s^{(k-1)})$ for all other $(k-1)$-blocks within $I_j^{(k)}$, we know there exists an open oriented path connecting the left boundary of $I_j^{(k)}$ to the right boundary of $I_{r-1}^{(k-1)}$ and an open oriented path connecting the left boundary of $I_{r+1}^{(k-1)}$ to the right boundary of $I_j^{(k)}$. These paths are obtained by concatenation of the corresponding $\Upsilon(I_s^{(k-1)})$, $s_0(j) \le s \le r-1$ and $r+1 \le s \le s_1(j)$, respectively.

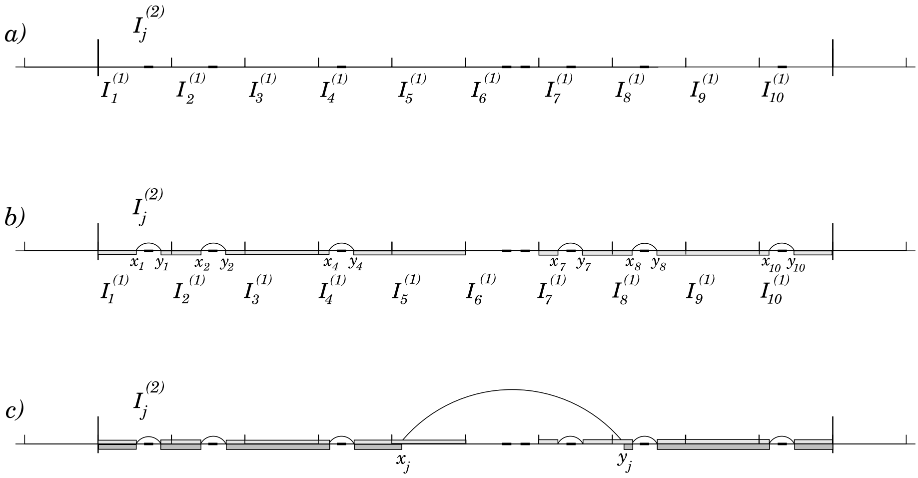

Fig. 2 Pedestals and defects: part (**a**) shows the deterministic 1-blocks $I_i^{(1)}$, $1 \le i \le 10$, located in the 2-block $I_j^{(2)}$; bold-face segments show location of the 0-defects. Part (**b**) shows construction of 1-pedestals, marked by light-gray strips. The block $I_6^{(1)}$ is a defected 1-block. The segments (x_i, y_i) are "enlarged" defects in $I_i^{(1)}$. Part (**c**) shows creation of 2-pedestals, marked by dark-gray strips, concatenated by long range edges. The segment (x_j, y_j) is the enlarged defect for $I_j^{(2)}$

$[a, a']$, with corresponding left and right pedestals $\Upsilon_{\mathcal{L}}$ and $\Upsilon_{\mathcal{R}}$, spanning from s to a and from a' to s', respectively, then

$$\left|\Upsilon_{\mathcal{L}} \cap [a - \lfloor l_k^{\alpha'/\alpha} \rfloor, a]\right| \ge C l_k^{\alpha'/\alpha} \quad \text{and} \quad \left|\Upsilon_{\mathcal{R}} \cap [a', a' + \lfloor l_k^{\alpha'/\alpha} \rfloor]\right| \ge C l_k^{\alpha'/\alpha}. \tag{2.8}$$

Inequality (2.8) follows trivially from the following recursive relation: if we have a k-block $I^{(k)} = \bigcup_{s_0}^{s_1} I_s^{(k-1)}$ which is in G state, then

$$\left|\Upsilon(I^{(k)})\right| \ge \sum_{s:\,[G(I_s^{(k-1)}) \text{ occurs}]} \left|\Upsilon(I_s^{(k-1)})\right| - l_k^{\alpha'/\alpha}.$$

We now give the announced basic estimate needed for the recursive step in the previous construction, cf. (1.8). Afterwards, we fix the parameters which will determine the choice of p close to one, as in (1.6). In the lemma below, assume that $I_j^{(k)}$ is a k-block and $D_j^{(k)} = \{z\}$, i.e. the unique defected $(k-1)$-block within $I_j^{(k)}$ has index z, and by construction stays at distance larger than $\lfloor l_k^{\alpha'/\alpha} \rfloor$ from the boundaries of $I_j^{(k)}$.

Lemma 2.1 *There exists $\eta \equiv \eta(\alpha, \alpha', l_1)$ with $\eta \searrow 0$ as $l_1 \nearrow +\infty$ and such that the following estimate for the conditional probability with respect to the product measure (defined right above (1.1))*

$$\begin{aligned} &\nu\big[\exists x \in \Upsilon_{\mathcal{L}}(I_j^{(k)}), y \in \Upsilon_{\mathcal{R}}(I_j^{(k)}),\ y - x \\ &\quad \le \lfloor l_k^{\alpha'/\alpha} \rfloor : \omega_{\{x,y\}} = 1 \big| |D_j^{(k)}| = 1\big] \ge 1 - l_{k-1}^{-\beta(1-\eta)(\alpha'-1)} \end{aligned} \tag{2.9}$$

holds for $k \ge 2$. For $k = 1$ the r.h.s in (2.9) is replaced by $1 - l_1^{-\beta(1-\eta)(\alpha'-1)/\alpha}$.

Proof We show the above estimate by conditioning on $D_j^{(k)}=\{z\}$, uniformly in z, and we make repeated use of the following upper and lower bounds: if I and I' are two intervals, and $3\le d=\mathrm{dist}(I,I')$, then

$$C^-\,J(I,I')\le\sum_{\substack{x\in I\cap\mathbb{Z}\\ y\in I'\cap\mathbb{Z}}}\frac{1}{|x-y|^2}\le C^+\,J(I,I'),\tag{2.10}$$

holds with $C^\pm=(1\pm 2/d)^2$ and

$$J(I,I')=\int_{I\times I'}dx\,dy\frac{1}{|x-y|^2}=\ln\frac{(|I|+d)(|I'|+d)}{d\,(|I|+|I'|+d)}.\tag{2.11}$$

Notice that we have $C^-\,(|x-y|-2)^{-2}\le|x-y|^{-2}\le C^+\,(|x-y|+2)^{-2}$ for $|x-y|\ge d$. We shall need also the inequality

$$J(I,I')\le 4\frac{|I'|}{|I''|}J\left(I,I''\right)\tag{2.12}$$

which holds for every I, I' and I'' such that $I'\subset I''$ and $d'=\mathrm{dist}(I,I'')\ge|I''|$. Indeed, setting $f(x)=\int_I dy\;|x-y|^{-2}$, for $x\in I''$, straightforward calculations give that under the above conditions:

$$f(x')\le 4f(x'')\quad\text{for each } x'\in I',\,x''\in I''$$

from where the inequality (2.12) follows upon integration.

If $k\ge 2$ and $D_j^{(k)}=\{z\}$, we have the left k-pedestal $\Upsilon_{\mathcal{L}}(I_j^{(k)})$ spanning from the left endpoint of $I_j^{(k)}$ to the left endpoint of $I_z^{(k-1)}$, and the right k-pedestal $\Upsilon_{\mathcal{R}}(I_j^{(k)})$, spanning from the right endpoint of $I_z^{(k-1)}$ to the right endpoint of $I_j^{(k)}$. Take two segments $S_z^{\mathcal{L}}$ and $S_z^{\mathcal{R}}$, such that $|S_z^{\mathcal{L}}|=|S_z^{\mathcal{R}}|=\lfloor\lfloor l_k^{\alpha'/\alpha}\rfloor/3\rfloor$, lying immediately to the left and, respectively, to the right of $I_z^{(k-1)}$. Denote

$$\widehat{\Upsilon}_{\mathcal{L}}(I_j^{(k)})=\Upsilon_{\mathcal{L}}(I_j^{(k)})\cap S_z^{\mathcal{L}},$$
$$\widehat{\Upsilon}_{\mathcal{R}}(I_j^{(k)})=\Upsilon_{\mathcal{R}}(I_j^{(k)})\cap S_z^{\mathcal{R}}.$$

Then

$$\begin{aligned}&\nu[\text{all edges }\{x,y\},\ x\in\widehat{\Upsilon}_{\mathcal{L}}(I_j^{(k)}),\ y\in\widehat{\Upsilon}_{\mathcal{R}}(I_j^{(k)})\text{ are closed}|\,D_j^{(k)}=\{z\}]\\&\le\prod_{\substack{x\in S_z^{\mathcal{L}}\\ y\in S_z^{\mathcal{R}}}}q_{\{x,y\}}\prod_{\substack{x\in S_z^{\mathcal{L}}\setminus\widehat{\Upsilon}_{\mathcal{L}}(I_j^{(k)})\\ y\in S_z^{\mathcal{R}}}}q_{\{x,y\}}^{-1}\prod_{\substack{x\in S_z^{\mathcal{L}}\\ y\in S_z^{\mathcal{R}}\setminus\widehat{\Upsilon}_{\mathcal{R}}(I_j^{(k)})}}q_{\{x,y\}}^{-1}.\end{aligned}\tag{2.13}$$

Applying (2.10) to $S_z^{\mathcal{L}}$ and $S_z^{\mathcal{R}}$ we immediately get the following bound:

$$\prod_{\substack{x\in S_z^{\mathcal{L}}\\ y\in S_z^{\mathcal{R}}}}q_{\{x,y\}}=\exp\left\{-\sum_{\substack{x\in S_z^{\mathcal{L}}\\ y\in S_z^{\mathcal{R}}}}\frac{\beta}{|x-y|^2}\right\}\le l_{k-1}^{-\beta(\alpha'-1)(1-b)},\tag{2.14}$$

where $b \equiv b(\alpha', l_1)$ and $b \searrow 0$ when $l_1 \nearrow +\infty$. Similar computation gives that if a 1-block I has a unique closed edge $\{a, a+1\}$ with both $a, a+1$ at distance larger than $l_1^{\alpha'/\alpha}$ from the endpoints of I, then the probability that there is an open edge $\{x, y\}$ with $x < a < y$, $y - x \leq l_1^{\alpha'/\alpha}$ is larger than or equal of $1 - l_1^{-\beta(1-\eta)(\alpha'-1)/\alpha}$.

On the other hand denoting by $\mathcal{D}_n(S_z^{\mathcal{L}} \setminus \widehat{\Upsilon}_{\mathcal{L}}(I_j^{(k)}))$, $0 \leq n \leq k-2$ (resp. $\mathcal{D}_n(S_z^{\mathcal{R}} \setminus \widehat{\Upsilon}_{\mathcal{R}}(I_j^{(k)})))$ the set of vertices that belong to all defected n-blocks contained in the segment $S_z^{\mathcal{L}}$ (resp. $S_z^{\mathcal{R}}$), we get

$$\begin{aligned}\prod_{\substack{x \in S_z^{\mathcal{L}} \setminus \widehat{\Upsilon}_{\mathcal{L}}(I_j^{(k)}) \\ y \in S_z^{\mathcal{R}}}} q_{\{x,y\}} &= \prod_{n=0}^{k-2} \prod_{\substack{x \in \mathcal{D}_n(S_z^{\mathcal{L}} \setminus \widehat{\Upsilon}_{\mathcal{L}}(I_j^{(k)})) \\ y \in S_z^{\mathcal{R}}}} q_{\{x,y\}} \\ &= \exp\left\{ - \sum_{n=0}^{k-2} \sum_{\substack{x \in \mathcal{D}_n(S_z^{\mathcal{L}} \setminus \widehat{\Upsilon}_{\mathcal{L}}(I_j^{(k)})) \\ y \in S_z^{\mathcal{R}}}} \frac{\beta}{|x-y|^2} \right\}.\end{aligned}$$

Once again, applying (2.10) for each $0 \leq n \leq k-2$ and taking into account the structure of n-pedestals together with (2.12), we have (uniformly on all l_1 large enough) fixed positive constants C_i, $i = 1, 2, 3$ so that

$$\begin{aligned}\sum_{n=0}^{k-2} \sum_{\substack{x \in \mathcal{D}_n(S_z^{\mathcal{L}} \setminus \widehat{\Upsilon}_{\mathcal{L}}(I_j^{(k)})) \\ y \in S_z^{\mathcal{R}}}} \frac{\beta}{|x-y|^2} &\leq C_1 \sum_{n=0}^{k-2} \sum_{\nu} J\left(I'_\nu, I^{\mathcal{R}}\right) \\ &\leq C_2 \sum_{n=0}^{k-2} \frac{l_{n+1}^{\alpha'/\alpha}}{l_{n+1}} \sum_{\nu} J\left(I''_\nu, I^{\mathcal{R}}\right) \\ &\leq C_3 l_1^{\alpha'/\alpha - 1} J\left(I^{\mathcal{L}}, I^{\mathcal{R}}\right)\end{aligned}$$

where I'_ν and I''_ν are intervals in $\mathbb{R}$ so that $\bigcup_\nu (I'_\nu \cap \mathbb{Z}) = \mathcal{D}_n(S_z^{\mathcal{L}} \setminus \widehat{\Upsilon}_{\mathcal{L}}(I_j^{(k)}))$, the sum $\sum_\nu$ is taken over all indices ν of $(n+1)$-blocks $I_\nu^{(n+1)} =: I''_\nu \cap \mathbb{Z}$ where the defected n-blocks are located, and moreover, $I^{\mathcal{L}} = \bigcup_{0 \leq n \leq k-2} \bigcup_\nu I''_\nu$ and $I^{\mathcal{R}}$ is the convex envelop of $S_z^{\mathcal{R}}$. The condition to apply (2.12) in the first inequality above follows from $3l_{k-1} + 6l_{k-2} \leq l_k^{\alpha'/\alpha}$ which is true for any $k \geq 2$, provided l_1 has been taken large enough. From this we can easily get that

$$\prod_{\substack{x \in S_z^{\mathcal{L}} \setminus \widehat{\Upsilon}_{\mathcal{L}}(I_j^{(k)}) \\ y \in S_z^{\mathcal{R}}}} q_{\{x,y\}} \geq l_{k-1}^{-\beta(\alpha'-1)b'}, \tag{2.15}$$

where $b' \equiv b'(\alpha, \alpha', l_1)$ and $b' \searrow 0$ when $l_1 \nearrow +\infty$. Analogous lower bound holds for the third term at the r.h.s of (2.13). Finally, from the upper bound for the length of a $(k-1)$-block, we have

$$\begin{aligned}&[\omega : \exists x \in \Upsilon_{\mathcal{L}}(I_j^{(k)}), y \in \Upsilon_{\mathcal{R}}(I_j^{(k)}),\ y - x \leq \lfloor l_k^{\alpha'/\alpha} \rfloor : \omega_{\{x,y\}} = 1]^c \\ &\quad \subseteq [\text{all edges } \{x,y\},\ x \in \widehat{\Upsilon}_{\mathcal{L}}(I_j^{(k)}),\ y \in \widehat{\Upsilon}_{\mathcal{R}}(I_j^{(k)}) \text{ are closed}],\end{aligned} \tag{2.16}$$

the statement of the Lemma follows from (2.14) and (2.15). □

Fixing the Parameters For fixed $\beta > 1$, which is the first main parameter of the model we choose the pair $\alpha,\ \alpha'$ with $1 < \alpha' < \alpha < 2$ such that

$$\beta(\alpha' - 1) - \frac{2(\alpha - 1)^2}{2 - \alpha} > \alpha - 1, \tag{2.17}$$

i.e. $\beta(\alpha' - 1) > \alpha\,(\alpha - 1)\,/(2 - \alpha)$. We also fix

$$\delta > \frac{2(\alpha - 1)}{2 - \alpha} \tag{2.18}$$

such that

$$\beta(\alpha' - 1) - \delta(\alpha - 1) > \alpha - 1. \tag{2.19}$$

By Lemma 2.1 we can fix $l_1 > 1$ so large that the parameter $\eta = \eta(\alpha, \alpha', l_1)$ in (2.9) becomes so close to zero, that

$$\beta(1 - \eta)(\alpha' - 1) - \delta(\alpha - 1) > \alpha - 1. \tag{2.20}$$

Inequalities (2.9), (2.18) and (2.20) are crucial for the inductive estimates.

Cluster of the Origin From the above estimates, and recalling (2.7), the initial heuristic discussion is indeed made rigorous: for the above choice of parameters and picking l_1 large enough we (recursively) obtain that for all $M \geq 1$ and at all scales $k = 1, \ldots, M$,

$$\nu(I_j^{(k)} \text{ is defected }) \leq l_k^{-\delta}. \tag{2.21}$$

Indeed, due to (2.7), we see that the previous analysis and the above choice of the parameters turns rigorous the discussion leading to (1.7) and (1.9). Now, for $k = M$, we have at most two M-blocks, denoted by $I_i^{(M)}$, where $1 \leq i \leq s$ and $s(\omega) \in \{1, 2\}$. In particular, from (2.21), we immediately have the basic estimate (1.4) announced in the introduction. Next we give the uniform lower bound for

$$\nu\left(0 \rightsquigarrow y,\ \text{for some } y \in \left[L - 2\lfloor l_M^{\alpha'/\alpha}\rfloor, L\right]\right).$$

Recalling that $j_z^k = \min\{j : z \in I_j^{(k-1)}\}$, for any $1 \leq k \leq M$ we define the following events:

$$\psi^{(k)} = \bigcap_{i=j_0^k - \lfloor\lfloor l_k^{\alpha'/\alpha}\rfloor/l_{k-1}\rfloor}^{j_0^k + \lfloor\lfloor l_k^{\alpha'/\alpha}\rfloor/l_{k-1}\rfloor} G(I_i^{(k-1)}) \tag{2.22}$$

and consider

$$\Psi_M = \bigcap_{j=1}^{M} \psi^{(j)}. \tag{2.23}$$

The occurrence of $\bigcap_{k=1}^n \psi^{(k)}$, $1 \leq n \leq M$ implies that the origin 0 is the right (resp. left) end-vertex of a $(n-1)$-block $I_{j_0^n}^{(n-1)}$ (resp. $I_{j_0^n+1}^{(n-1)}$) for each n, since no adjustments are performed in this case, and necessarily it belongs to the pedestals $\Upsilon(I_{j_0^n}^{(n-1)})$ and $\Upsilon(I_{j_0^n+1}^{(n-1)})$

for any $1 \le n \le M$. In particular, for $\omega \in \Psi_M$ we have $s(\omega) = 2$. Moreover, in the event $\Psi_M \cap G(I_1^{(M)}) \cap G(I_2^{(M)})$, the origin 0 belongs to an open oriented path connecting $[-L, -L + 2\lfloor l_M^{\alpha'/\alpha} \rfloor]$ to $[L - 2\lfloor l_M^{\alpha'/\alpha} \rfloor, L]$ as described above.
Taking into account the estimate (2.21) and the definition (2.22) we have for $k \ge 2$:

$$\nu(\psi^{(k)}) \ge 1 - (2(l_{k-1})^{\alpha'-1} + 1)(l_{k-1})^{-\delta} \ge 1 - 3(l_{k-1})^{\alpha'-1}(l_{k-1})^{-\delta}.$$

Since

$$\delta - (\alpha' - 1) > \delta - (\alpha - 1) \ge \frac{\alpha(\alpha - 1)}{2 - \alpha} > 0,$$

we define $u = \delta - (\alpha' - 1) > 0$ and rewrite the above inequality:

$$\nu(\psi^{(k)}) \ge 1 - 3(l_{k-1})^{-u} \quad \text{for } k \ge 2.$$

Since l_k grow super-exponentially fast, we get immediately that the series

$$(l_1)^{-u} + (l_2)^{-u} + (l_3)^{-u} + \cdots = S(l_1)$$

converges and

$$S(l_1) \longrightarrow 0, \quad \text{when } l_1 \to \infty.$$

This immediately implies that

$$\nu\left(\left[\bigcap_{j=2}^{M} \psi^{(j)} \cap G(I_1^M) \cap G(I_2^M)\right]^c\right) \tag{2.24}$$

can be made arbitrarily small, uniformly in M.
Finally, by choosing l_1 large enough, and then p close enough to 1 we get that $\nu(\psi^{(1)})$ can be made arbitrarily close to 1.
The proof of Theorem 1.1 follows at once. □

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